NOTE ON A THEOREM REGARDING A SUM OF DIFFERENTIAL-COEFFICIENTS OF PRINCIPAL MINORS OF A JACOBIAN.

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1. The theorem in question is that which plays so important a part in Jacobi's method* of determining the last integrating factor for a set of differential equations, and which he accordingly styled his "fundamental lemma." It may be enunciated in simple manner as follows: *If* $A_1, A_2, \ldots, A_n$ *be the co-factors of the elements of any row of a Jacobian whose independent variables are* $x_1, x_2, \ldots, x_n$, *then*

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \ldots\ldots + \frac{\partial A_n}{\partial x_n} = 0.$$

2. Of this theorem in 1854 Donkin gave a very peculiar demonstration.† Separating the symbols of operation

$$\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \ldots\ldots$$

from the subjects operated on, which in the case of $n = 4$ may be written

$$\begin{vmatrix} \frac{\partial u}{\partial x_2} & \frac{\partial v}{\partial x_3} & \frac{\partial w}{\partial x_4} \end{vmatrix}, \quad -\begin{vmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial v}{\partial x_3} & \frac{\partial w}{\partial x_4} \end{vmatrix}, \ldots\ldots$$

he was, of course, enabled to put the left-hand member in the form

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} \\ \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial x_3} & \frac{\partial u}{\partial x_4} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} & \frac{\partial v}{\partial x_3} & \frac{\partial v}{\partial x_4} \\ \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & \frac{\partial w}{\partial x_3} & \frac{\partial w}{\partial x_4} \end{vmatrix}.$$

* Jacobi, C. G. J., Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi. *Crelle's Journal*, xxvii., pp. 169–268; xxix., pp. 213–279, 333–375.

† Donkin, W. F., Demonstration of a theorem of Jacobi relative to functional determinants. *Cambridge and Dub. Math. Journ.*, ix., pp. 161–163.

He then contrived to satisfy himself that for the first element of each column he was entitled to substitute an expression which turned out to be the sum of the remaining elements of the column. The vanishing of the determinant was thus concluded to be inevitable.

3. Now it is readily seen that in effect this was equivalent to saying that the determinant could be expressed as the sum of $n-1$ determinants of the nth order, that is to say, as an aggregate of $(n-1)(n!)$ terms of the nth degree in the first differential-coefficients of $u, v, \ldots$; whereas A_1 being an aggregate of $(n-1)!$ terms of the $(n-1)$th degreee in the said differential-coefficients, $\frac{\partial A_1}{\partial x_1}$ must be an aggregate of $(n-1)!(n-1)!$ terms in each of which a *second* differential-coefficient occurs, and therefore $\Sigma\frac{\partial A_1}{\partial x_1}$ must be an aggregate of $(n-1)!\,n!$ terms of this latter kind. Such being the case, doubt is at once thrown on the so-called demonstration. A little examination of the reasoning employed fully justifies us in setting it aside as misleading and fallacious.

4. Fortunately our scepticism leads us at the same time to produce as a substitute a valid proof not hitherto made known.

Recalling the fact that the differential-coefficient of an n-line determinant can be expressed as the sum of n determinants of the same order, we see that $\Sigma\frac{\partial A_1}{\partial x_1}$ can be expressed as an array of n rows with $n-1$ determinants in each row. If then the elements of each column in this array be added, it will be found that the result is zero, in accordance with a well-known theorem of Kronecker's regarding vanishing aggregates of determinants.

Thus, when $n = 4$, we have on writing u_{rs} for $\partial^2 u/\partial x_r \partial x_s$

$$\frac{\partial A_1}{\partial x_1} = \begin{vmatrix} u_{12} & u_{13} & u_{14} \\ v_2 & v_3 & v_4 \\ w_2 & w_3 & w_4 \end{vmatrix} + \begin{vmatrix} u_2 & u_3 & u_4 \\ v_{12} & v_{13} & v_{14} \\ w_2 & w_3 & w_4 \end{vmatrix} + \begin{vmatrix} u_2 & u_3 & u_4 \\ v_2 & v_3 & v_4 \\ w_{12} & w_{13} & w_{14} \end{vmatrix},$$

$$\frac{\partial A_2}{\partial x_2} = -\begin{vmatrix} u_{21} & u_{23} & u_{24} \\ v_1 & v_3 & v_4 \\ w_1 & w_3 & w_4 \end{vmatrix} - \begin{vmatrix} u_1 & u_3 & u_4 \\ v_{21} & v_{23} & v_{24} \\ w_1 & w_3 & w_4 \end{vmatrix} - \begin{vmatrix} u_1 & u_3 & u_4 \\ v_1 & v_3 & v_4 \\ w_{21} & w_{23} & w_{24} \end{vmatrix},$$

$$\frac{\partial A_3}{\partial x_3} = \begin{vmatrix} u_{31} & u_{32} & u_{34} \\ v_1 & v_2 & v_4 \\ w_1 & w_2 & w_4 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_4 \\ v_{31} & v_{32} & v_{34} \\ w_1 & w_2 & w_4 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_4 \\ v_1 & v_2 & v_4 \\ w_{31} & w_{32} & w_{34} \end{vmatrix},$$

$$\frac{\partial A_4}{\partial x_4} = -\begin{vmatrix} u_{41} & u_{42} & u_{43} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} - \begin{vmatrix} u_1 & u_2 & u_3 \\ v_{41} & v_{42} & v_{43} \\ w_1 & w_2 & w_3 \end{vmatrix} - \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_{41} & w_{42} & w_{43} \end{vmatrix},$$

and therefore by addition

$$\Sigma\frac{\partial A_1}{\partial x_1} = 0 + 0 + 0.$$

5. When we recall that Kronecker's theorem here used was discovered by him as a property of minors of an *axisymmetric* determinant, and that indeed it has not yet been formulated apart from this connection, the interest in the demonstration is greatly heightened. For example, to assert the vanishing of the first column of determinants in the preceding paragraph is the same as to say in reference to the axisymmetric determinant

$$\begin{vmatrix} \cdot & \cdot & v_1 & v_2 & v_3 & v_4 \\ \cdot & \cdot & w_1 & w_2 & w_3 & w_4 \\ v_1 & w_1 & \cdot & u_{12} & u_{13} & u_{14} \\ v_2 & w_2 & u_{12} & \cdot & u_{23} & u_{24} \\ v_3 & w_3 & u_{13} & u_{23} & \cdot & u_{34} \\ v_4 & w_4 & u_{14} & u_{24} & u_{34} & \cdot \end{vmatrix}$$

that

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{vmatrix} = 0,$$

where $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$ stands for the three-line minor whose elements are to be found in the 1st, 2nd, 3rd rows and in the 4th, 5th, 6th columns.